Stability Analysis of All Possible Equilibria for Gyrostat Satellites under Gravitational Torques

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Introducing a source of angular momentum into a gravitationally stabilized satellite affects both the equilibria and the stability properties. A previous paper found that there is a two parameter family of orientations which can be made into equilibria, relative to gravitational torques, by using a source of internal angular momentum. Here a stability analysis is made of all possible equilibria for such gyrostat satellites. The parameters involved are the three inertias of the satellite, two parameters identifying the equilibrium, and the component of angular momentum perpendicular to the orbital plane. The boundaries in this parameter space along which the qualitative nature of the stability behavior changes, are identified using the theory of bifurcation of Poincaré with the quadratic approximation to the dynamic potential. The region of Liapunov stable satellites (using the Hamiltonian as a Liapunov function) is identified. It is shown that the case 4a solutions previously described (having the axis of intermediate moment of inertia any place in the orbital plane) forms a boundary dividing those satellites which can be made Liapunov stable from those which cannot be made Liapunov stable by choosing a sufficiently large component of internal angular momentum along the perpendicular to the orbital plane. This boundary can be easily visualized. Contour plots show the magnitude of this angular momentum component required to obtain stability.

Introduction

GRAVITATIONALLY stabilized satellites are designed to use gravitational torques to keep one side of the satellite pointing toward the Earth. It has long been recognized that introducing a source of angular momentum (for example, a constant speed symmetric rotor) into such a satellite can significantly improve its attitude stability properties. To date most of the attention has been focused on the elementary case where the internal angular momentum vector is aligned with a principal axis of the satellite, and that axis in turn is aligned with the perpendicular to the orbital plane (the dual-spin spacecraft that have been orbited could be considered to fall into this category). Several authors have investigated special cases of what happens if the angular momentum is not aligned with a principal axis (see Refs. 1–5). Such a configuration causes a somewhat different side of the satellite to face toward the Earth when the satellite is in equilibrium relative to gravitational torques, and of course the stability properties of the equilibria are also altered. The approach has been to select the angular momentum vector or a set of angular momentum vectors, then find some or all of the associated satellite equilibrium orientations and determine whether they are stable or unstable. Those that have the desired stability properties might be employed in actual satellite designs. It was shown in a previous paper[6] that a panoramic view of all the possibilities involved can be obtained by adopting a different approach, and asking what is the set of satellite orientations which can be made into equilibrium orientations by proper choice of an internal angular momentum vector within the satellite. This set of orientations was found to form a two parameter family which can easily be visualized. Since this approach identifies all the possible equilibria it is of considerable interest to make a study of stability from this point of view. This study is the subject of the present paper. The stability analysis is complicated by the number of parameters involved: the three principal moments of inertia of the satellite, the two parameters identifying any particular orientation within the set of all possible orientations, and a sixth parameter which is the component of the internal angular momentum vector along the perpendicular to the orbital plane (this component does not affect the equilibria but does affect the stability). By suitable normalization the number of parameters can be reduced from six to four and a good picture can be obtained showing how the stability region varies as a function of these parameters. Stability regions are determined by examining the properties of the dynamic potential in a neighborhood of the equilibria. Orientations which are not established as stable by this method might still be infinitesimally stable, but their stability would disappear in the presence of damping.

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Background

We assume that the satellite is in a circular orbit in an inverse square gravitational field. Equilibria generally do not exist otherwise. We also assume that the attitude of the satellite has no effect on the orbit of the satellite (Ref. 6 gives a description of the negligibly small effect occurring under equilibrium conditions). The satellite orientation is specified relative to an orbiting reference frame (sometimes called the local-vertical reference frame) with origin at the satellites' center of mass. The vector ξ_3 is a unit vector in the direction of the geocentric vertical, ξ_2 is normal to the plane of the orbit with the positive sense of the orbital angular velocity vector, and ξ_1 is along the forward tangent to the orbit. Equilibrium orientations are then satellite orientations which will remain unaltered in time, relative to the orbiting reference frame, in the presence of gravitational torques.

The internal angular momentum of the satellite is assumed to be due to the rotation of a symmetric rotor so that the inertia matrix I of the composite satellite is constant when expressed in any coordinate system fixed in the body. We define such a coordinate system with origin at the center of mass, axes along the principal axes, and with unit vectors along these axes denoted by $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$. Then these unit vectors are related to ξ_1, ξ_2, ξ_3 by an orthogonal direction cosine matrix Θ

$$\mathbf{X}_\alpha = \sum_{\beta=1}^{3} \Theta_{\alpha\beta}\,\xi_\beta \qquad \alpha = 1,2,3$$

We define 3 by 1 matrices ξ_1, ξ_2, ξ_3 which are components of ξ_1, ξ_2, ξ_3 along body axes. The Θ matrix can then be partitioned as $[\xi_1 \quad \xi_2 \quad \xi_3]$. Also ω is a 3 by 1 matrix representing the angular velocity of the body in inertial space expressed in body coordinates, and h is the matrix of components along body axes of the relative angular momentum of the rotors with respect to body axes. We further define ω_0 to be the angular velocity of a point mass in the given circular orbit in an inverse square gravitational field. We let J_1, J_2, and J_3 be components of h/ω_0 along ξ_1, ξ_2, and ξ_3, respectively. Reference 6 transforms the conditions for equilibrium relative to gravitational torques into the following six equations

$$\begin{aligned} &\xi_1{}^T I \xi_1 = \lambda, \quad \xi_2{}^T I \xi_1 = -J_1, \quad \xi_2{}^T I \xi_2 = \mu \\ &\xi_3{}^T I \xi_1 = 0, \quad \xi_3{}^T I \xi_3 = \nu, \quad \xi_3{}^T I \xi_2 = -\tfrac{1}{4} J_3 \end{aligned} \tag{1}$$

Of course the ξ_α must also satisfy the orthonormality conditions

$$\xi_\alpha{}^T \xi_\beta = \delta_{\alpha\beta} \tag{2}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The superscript T indicates transpose. The λ, μ, and ν above are arbitrary constants which we take to be defined by Eq. (1). Furthermore, since we are first picking the satellite orientation (i.e., picking ξ_1, ξ_2, ξ_3) and then determining the necessary internal angular momentum, the quantities J_1 and J_3 are in no way restricted a priori and therefore they are also determined by Eq. (1). (Note that J_2, the normalized component of internal angular momentum along the perpendicular to the orbital plane, has no effect on the equilibrium conditions). Thus, the only restriction on the satellite orientations which can be made into equilibria is the condition that $\xi_3{}^T I \xi_1$ be zero. This result was also obtained by another author and published the same year.[7] A physical interpretation of this condition can be obtained by recognizing that Eq. (1) represents the elements of the inertia matrix when represented in the ξ frame by the congruence relation $\Theta^T I \Theta$. Thus, any satellite orientation which preserves the zero of the 1, 3 product of inertia in the ξ frame, can be made into an equilibrium by proper choice of J_1 and J_3.

Family of Equilibrium Orientations

The set of all possible equilibrium orientations can be characterized as follows: The line connecting the center of mass of the satellite with the center of the gravitational force field can be chosen arbitrarily relative to body axes; that is, any side of the satellite can be made to face toward the Earth. Once this is chosen there are precisely two possible orientations of the satellite which can be made into equilibria by proper choice of internal angular momentum. Thus, we can choose the direction of ξ_3 (the geocentric vertical) arbitrarily relative to body axes. Let us specify this direction relative to body axes by two polar coordinates ϕ and ψ as shown in Fig. 1. In the present approach these two angles are the two parameters of the two parameter family of equilibria. Let θ and η be polar coordinates of the unit vector ξ_1 as shown in Fig. 1. Then

$$\xi_1 = \begin{bmatrix} \cos\eta\cos\theta \\ \cos\eta\sin\theta \\ \sin\eta \end{bmatrix} \qquad \xi_3 = \begin{bmatrix} \cos\psi\cos\phi \\ \cos\psi\sin\phi \\ \sin\psi \end{bmatrix} \tag{3}$$

and ξ_2 can be obtained by a cross product. The equation $\xi_3{}^T I \xi_1 = 0$ becomes

$$\tan\theta = [(I_3 - I_1)/(I_2 - I_3)]\cot\phi \tag{4}$$

Without loss of generality we take $\mathbf{X}_3$ to be the axis of intermediate moment of inertia so that the coefficient above is positive. Then as ϕ increases from zero to 90°, θ goes from 90° to zero. The θ obtained from the above equation specifies the plane containing ξ_1. Examination of Fig. 1 shows that there is usually only one line in this plane that is perpendicular to ξ_3. We can choose the positive sense of ξ_1 to be either direction along this line thus specifying two possible equilibrium orientations. The angle η can be obtained by use of the law of cosines for spherical triangles

$$\tan(\eta) = -\cot(\psi)\cos(\theta - \phi) \tag{5}$$

Given any ϕ and ψ we obtain θ and η from Eqs. (4) and (5).

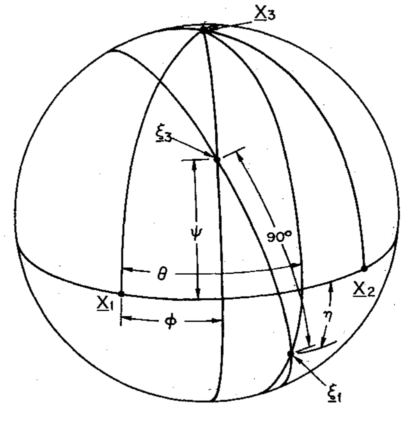

Fig. 1 Geometric description of all equilibrium orientations.

Then the required internal angular momentum vector can be obtained by using Eqs. (3) in Eq. (1) to obtain J_1 and J_3 (J_2 remains arbitrary).

The above geometrical visualization of the two parameter family has several degenerate cases. They arise when θ and ϕ from Eq. (4) differ by 90°, that is when $\phi = 0$ or 90°. One example occurs when ϕ and ψ are both zero so that any vector in the $\theta = 90°$ plane will be perpendicular to ξ_3, and η can take on all values. These degeneracies are the special case solutions which have already been treated by several authors (Refs. 1–5). It is important to see how they relate to the general solution, so we include the definitions here: 1) The principal axes of the body aligned in any way with $\pm\xi_1$, $\pm\xi_2$, and $\pm\xi_3$. Both J_1 and J_3 must be zero; 2) A principal axis aligned with $\pm\xi_1$, and the other body axes not aligned with $\pm\xi_2$, $\pm\xi_3$. This requires $J_1 = 0$, $J_3 \neq 0$; and 3) A principal axis aligned with $\pm\xi_3$, and the other body axes not aligned with $\pm\xi_1$, $\pm\xi_2$. Here $J_1 \neq 0$, $J_3 = 0$.

Stability analyses of these special cases have appeared in the literature (Refs. 1–5). The general case was split into two subcases in Ref. 6: 4a) The axis of intermediate moment of inertia in the orbital plane and none of the principal axes aligned with an axis of the ξ frame. This case arises when $\phi = \theta$ in Fig. 1. $J_1 \neq 0$, $J_3 \neq 0$; and 4b) None of the principal axes in any of the planes: $\xi_1 - \xi_2$ plane, $\xi_1 - \xi_3$ plane or the $\xi_2 - \xi_3$ plane. $J_1 \neq 0$, $J_3 \neq 0$.

It will be shown that case 4a divides those satellites which are stable for a sufficiently large angular momentum component along ξ_2, from those satellites which cannot be stabilized by increasing J_2.

Reference 6 showed that the above classification is complete so that all possible equilibria are included. Furthermore, it was shown that there are no case 4 solutions for inertially symmetric satellites.

Stability Theory

We assume that the internal angular momentum is a constant vector within the satellite, which presupposes a source of energy to maintain the speed of the rotor. In spite of this, it can be established that we still have a conservative system so that the Hamiltonian

$$H = \tfrac{1}{2}\omega^T I \omega + \tfrac{3}{2}\omega_0{}^2 \xi_3{}^T I \xi_3 - \omega_0 \xi_2{}^T h - \omega_0 \xi_2{}^T I \omega$$

is a constant of the motion.[8] Note that since this is a gyroscopic system with the generalized coordinates referenced to a rotating reference frame, the Hamiltonian will not be the total energy of the system. It can be expressed as

$$H = T_2 + U = T_2 + (V - T_0)$$

where T_2 is that part of the kinetic energy which is quadratic in the generalized velocities and T_0 is that part which is independent of the generalized velocities. The function U is the dynamic potential (V is the pontential energy).

Let us describe any deviation from an equilibrium orientation by 1, 2, 3 type Euler angles (Tait-Bryan angles). Thus, θ_1 is a rotation about the equilibrium $\mathbf{X}_1$ body axis, which is followed by a rotation θ_2 about the resulting $\mathbf{X}_2$ body axis and then a rotation of angle θ_3 about the resulting $\mathbf{X}_3$ axis to arrive at the actual body axes. We expand the Hamiltonian in a Taylor series about any equilibrium orientation. We are free to add a constant to the Hamiltonian so that any constant term arising in the expansion can be neglected. The coefficients of any linear terms in the expansion will automatically be zero since the expansion is taken about an equilibrium orientation. Through quadratic terms the Hamiltonian can be written as

$$H = \tfrac{1}{2}\dot{\theta}_e{}^T I \dot{\theta}_e + \tfrac{1}{2}\omega_0{}^2 \theta_e{}^T \mathscr{K} \theta_e \tag{6}$$

where

$$\mathscr{K} = [\tilde{\xi}_2 I - (I\xi_2)^{\sim} - \tilde{h}/\omega_0]\tilde{\xi}_2 - 3[\tilde{\xi}_3 I - (I\xi_3)^{\sim}]\tilde{\xi}_3 \tag{7}$$

(see Ref. 8). Here θ_e represents the matrix $\theta_e{}^T = [\theta_1 \quad \theta_2 \quad \theta_3]$ and the tilde symbol over any column vector, say θ_e, denotes the square antisymmetric matrix whose components are $\varepsilon_{\alpha\lambda\beta}\theta_\lambda$ ($\varepsilon_{\alpha\lambda\beta}$ is the Levi-Civita density, or "epsilon symbol" of tensor analysis). The term $\frac{1}{2}\omega_0{}^2\theta_e{}^T\mathscr{K}\theta_e$ is the quadratic approximation to the dynamic potential, so that $\mathscr{K}$ is, to within a constant multiple, the Hessian matrix of the dynamic potential. The matrices ξ_2 and ξ_3 must be evaluated at the equilibrium which establishes $\mathscr{K}$ as a symmetric matrix.

Since the Hamiltonian is a constant of the motion it can be used as a Liapunov function in the basic Liapunov stability theorem.[9] Thus, a sufficient condition for an equilibrium orientation to be Liapunov stable is that the Hamiltonian represent a positive definite function in a region about the equilibrium in the phase space. Since all variables of the phase space are present in the quadratic approximation, and since the inertia matrix is necessarily positive definite, we see that a sufficient condition for stability is that the Hessian matrix be positive definite. If this condition is not satisfied the equilibrium may be infinitesimally stable due to gyroscopic effects. The boundaries of the region of positive definiteness in the parameter space will be identified using bifurcation theory.[10]

The above assumes that the satellite is rigid except for the internal spinning rotor. In any physical system there will be energy dissipation. Suppose the damping is due to forces which do not affect the mass distribution of the satellite (e.g., magnetic losses due to the Earth's field) or to forces of internal damping which involve internal elements with small mass and inertia. Suppose further that the dissipation is such that the satellite is completely damped so that the time derivative of the Hamiltonian can be identically zero for all time only if the satellite is in equilibrium. Thus, we suppose that the dissipative mechanism has negligible effect on the motion of the satellite except insofar as it gradually dissipates the energy of the satellite when it is not in equilibrium. Then we can apply the following theorem due to Pringle.[11]

Theorem 1: The equilibrium solution $\theta_i = 0$, $i = 1, 2, 3$ of a completely damped mechanical system is 1) asymptotically stable if U is positive definite in a region about the origin, and 2) unstable if U can take on negative values arbitrarily close to the origin.

Thus in the presence of damping the sufficient condition for stability, that $\mathscr{K}$ be positive definite, also becomes a necessary condition, and stability due to gyroscopic coupling disappears. Throughout this paper the term stable will be used to mean Liapunov stable with the Hamiltonian as a Liapunov function, and the term unstable will be used to mean unstable in the presence of damping as described above. Of course applying Theorem 1 to our satellite is not mathematically rigorous since the actual form of the dissipative mechanism is not included in our model, but it is nevertheless quite useful in an engineering sense.

Stability Analysis

If the $\mathscr{K}$ matrix of the quadratic form in θ_e is positive definite then the equilibrium used to evaluate $\mathscr{K}$ gives a local minimum to the dynamic potential. If it is negative definite the equilibrium gives a local maximum to the dynamic potential, and if $\mathscr{K}$ is sign variable U has a three dimensional "saddle point." If we vary the satellite design parameters and an eigenvalue changes sign, then the qualitative nature of the stability behavior changes, i.e., the topology of the dynamic potential changes. Since the determinant of a matrix equals the product of the eigenvalues, anytime an eigenvalue goes through zero the determinant of the Hessian matrix will also. This gives a simple test to determine boundaries of regions with different types of stability behavior. However, there is no guarantee that just because the determinant becomes zero, an eigenvalue will actually change sign. So some care must be exercized. The satellite design parameters are the three inertias I_1, I_2, I_3, the two parameters of the equilibrium family ϕ, ψ, and the angular momentum component J_2. Thus, the task at hand is to identify in a meaningful way the surfaces in the six dimensional parameter space along which the Hessian determinant is zero, and then determine what type of stability behavior lies on each side of these surfaces.

The components of the $\mathscr{K}$ matrix obtained from Eq. (7) appear hopelessly complicated. The matrix takes on a much simpler appearance if we express it in the ξ frame rather than body coordinates. This is accomplished by making the orthogonal similarity transformation $\Theta^T\mathscr{K}\Theta$. Of course the eigenvalues are unaltered in the process. From the definitions of h and J_α we can write

$$h/\omega_0 = J_1\xi_1 + J_2\xi_2 + J_3\xi_3$$

which we substitute into $\mathscr{K}$. By using the partitioned form of Θ it is easily seen that the elements of the transformed matrix are $\xi_\alpha{}^T\mathscr{K}\xi_\beta$. We write out each element and make frequent use of the following cross product relations: $\tilde{\xi}_\alpha\xi_\beta = -\tilde{\xi}_\beta\xi_\alpha = \xi_\gamma$ for α, β, γ any cyclic permutation of 1, 2, 3; the transpose of the preceding remembering $\tilde{\xi}_\alpha$ is skew-symmetric; $\tilde{\xi}_\alpha\xi_\alpha = [0]$; and $\tilde{a}b = -\tilde{b}a$ for any column matrices a and b. We also make use of Eq. (1) and obtain

$$\Theta^T\mathscr{K}\Theta = \begin{bmatrix} 4(\mu-\nu)+J_2 & 3J_1 & 0 \\ 3J_1 & 3(\lambda-\nu) & -3J_3/4 \\ 0 & -3J_3/4 & (\mu-\lambda)+J_2 \end{bmatrix} \tag{8}$$

Let us pick the body axes so that $\mathbf{X}_2$ is the axis of greatest moment of inertia, and $\mathbf{X}_1$ is the axis of least moment of inertia. This is consistent with the choice made below Eq. (4). Thus, the inertias have the ordering $I_2 \geqq I_3 \geqq I_1$. As mentioned before the family of equilibria degenerate to the special case solutions if any two of these inertias are equal. Now let us divide each element of the above matrix by the quantity $I_2 - I_1$. This quantity will be zero only if $I_1 = I_2 = I_3$, which represents a degenerate case of no interest to us. Note that this normalization has the effect of dividing each of the eigenvalues by $I_2 - I_1$, but since this factor is positive the signs of the eigenvalues are not changed. Let us denote the transformed and normalized $\mathscr{K}$ by $\mathscr{K}'$, and use a prime on λ, μ, ν, J_1 and J_3 to denote these quantities divided by $I_2 - I_1$. We define a new inertia parameter

$$K = (I_2 - I_3)/(I_2 - I_1)$$

Note that for the inertia ordering chosen we have $0 \leqq K \leqq 1$. Examination of Eq. (4) shows that the coefficient can be written as $(1-K)/K$ so that the equilibrium orientations depend only on this one inertia parameter. Furthermore use of the orthonormality conditions of Eq. (2) in Eq. (1) show that J_1' and J_3' can also be expressed as functions of only one inertia parameter K. Although λ, μ, and ν divided by $I_2 - I_1$ do not depend solely on this one inertia parameter, taking the difference between any two of them eliminates explicit dependence on I_2 leaving only dependence on K. Thus we see that each term of Eq. (8) divided by $I_2 - I_1$ will depend inertially only on K except for the terms J_2. We define a new parameter p as

$$p = J_2/(I_2 - I_1)$$

Then instead of our original set of six parameters, by suitable normalization we have been able to reduce the set to the following four: K, ϕ, ψ, p.

Let us examine the changes in the stability behavior as we vary p, the normalized component of internal angular momentum along the perpendicular to the orbital plane. One result can be seen easily from the form of Eq. (8). If p is changed by an amount Δp then the corresponding change in the matrix $\mathscr{K}'$ is

$$\Delta\mathscr{K}' = \begin{bmatrix} \Delta p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta p \end{bmatrix}$$

If Δp is positive (negative), $\Delta\mathscr{K}'$ is positive (negative) semidefinite. The sum of a positive (negative) definite matrix and a positive (negative) semidefinite matrix is still positive (negative) definite. This implies that if the satellite is stable for a given set of parameters it will always remain stable if we increase J_2. We state the result as a Lemma which we will make use of later: *Lemma 1*: If for a given p all the eigenvalues of $\mathscr{K}'$ are positive, then they will all be positive for any larger value of p. Furthermore, if for a given p all the eigenvalues of $\mathscr{K}'$ are negative, then they will all be negative for any smaller value of p. Further insight can be obtained by making use of two theorems due to Gerschgorin.[12]

Theorem 2: Every eigenvalue of the matrix $A = [a_{\alpha\beta}]$ lies in at least one of the circular disks with centers $a_{\alpha\alpha}$ and radii $\sum_{\alpha\neq\beta} |a_{\alpha\beta}|$.

Theorem 3: If s of the circular disks form a connected domain which is isolated from the other disks, then there are precisely s eigenvalues of A within this domain.

Our matrix is symmetric so that all the eigenvalues are real. Therefore the disks above can be replaced by intervals on the real line. We use these theorems to obtain the following result: *Lemma 2*: As p approaches $\pm\infty$, the eigenvalues of $\mathscr{K}'$ approach the diagonal elments $4(\mu'-\nu')+p$, $3(\lambda'-\nu')$, and $(\mu'-\lambda')+p$.

To prove this, note that the eigenvalues of a matrix are unaffected by multiplying a row by a constant ε, and the corresponding column by $1/\varepsilon$. We multiply the ith row by ε_i and ith column by $1/\varepsilon_i$ for all $i = 1, 2, 3$. Then any eigenvalue α must lie in at least one of the intervals

$$|\alpha - [4(\mu'-\nu')+p]| \leqq 3|\varepsilon_1 J_1/\varepsilon_2|$$
$$|\alpha - 3(\lambda'-\nu')| \leqq 3|\varepsilon_2 J_1/\varepsilon_1| + (\tfrac{3}{4})|\varepsilon_2 J_3/\varepsilon_3|$$
$$|\alpha - [(\mu'-\lambda')+p]| \leqq (\tfrac{3}{4})|\varepsilon_3 J_3/\varepsilon_2|$$

First let us take $\varepsilon_2 = 1$. Then given any ε_1 and ε_3 there will be a p sufficiently large that the first and third intervals will be separated from the second. Furthermore, if $4(\mu'-\nu') \neq (\mu'-\lambda')$ we can pick ε_1 and ε_3 so that the first and third intervals are disjoint. Then by Theorem 3 there will be one eigenvalue in each interval. Since we can make ε_1 and ε_3 as small as we like, we conclude that one of the eigenvalues is arbitrarily close to $4(\mu'-\nu')+p$, and one is arbitrarily close to $(\mu'-\lambda')+p$ for sufficiently large p. If $4(\mu'-\nu') = (\mu'-\lambda')$ two eigenvalues are arbitrarily close to $(\mu'-\lambda')+p$. Now let $\varepsilon_1 = \varepsilon_3 = 1$. Then, given any ε_2 no matter how small we can again find a p sufficiently large that the second interval is disjoint from the first and third. Thus one eigenvalue is arbitrarily close to $3(\lambda'-\nu')$ for sufficiently large p. Since all of the above arguments also hold if we let p tend toward $-\infty$, the Lemma is proved.

Lemma 2 has the following interpretation. If $\lambda' > \nu'$, then for large negative p, $\mathscr{K}'$ has two negative and one positive eigenvalue, and for large positive p, $\mathscr{K}'$ has three positive eigenvalues. If $\lambda' < \nu'$, then large negative p gives three negative eigenvalues, large positive p gives two positive and one negative eigenvalue. Thus we see that there must be at least two sign changes in the eigenvalues as p goes from $-\infty$ to $+\infty$. Note that when $\lambda' = \nu'$ we can no longer determine by this method the sign of the eigenvalue which approaches $3(\lambda'-\nu')$ since now all we know is that for sufficiently large p this eigenvalue is in an arbitrarily small neighborhood of the origin. This gives no information about the sign. Note also that a combination of the two Lemmas shows that only orientations for which $\lambda' > \nu'$ can be stabilized (in the sense described in the previous section) by use of internal angular momentum. It should be mentioned that the eigenvalues of $\mathscr{K}$ have an interesting physical interpretation. Consider an equilibrium orientation at which H has a local minimum. For any satellite with a slightly larger constant value of the Hamiltonian, call it c, the equation $U = c$ defines a closed "contour" about the equilibrium. This closed contour represents a bound on the deviation of the satellite orientation from equilibrium. If c is small enough the quadratic approximation to U will be a good representation of U and the equation $\frac{1}{2}\omega_0^2\theta_e^T\mathscr{K}\theta_e = c$ will approximate the bounds on the satellite librations. Thus we see that the larger the eigenvalues of $\mathscr{K}$, the tighter the bound becomes, at least in a local sense. Stating this another way, if the eigenvalues are increased, then it takes a larger energy change in the Hamiltonian to produce a given amount of maximum deviation from equilibrium when the velocities $\dot{\theta}_e$ are allowed to go to zero. Lemma 2 shows that by increasing p, two of the eigenvalues can be made arbitrarily large, but increasing p has only a limited effect on the third eigenvalue. In other words, for sufficiently large p we can get arbitrarily tight bounds for attitude changes in two directions but the bound on librations in the third direction [along the eigenvector of $\mathscr{K}'$ for the eigenvalue which approaches $3(\lambda'-\nu')$] cannot be made arbitrarily small by increasing p. For an equilibrium orientation where U is a local maximum the above methods give a bound on how close the satellite can come to the equilibrium for any given value of H. When U has a "saddle point" $U = c$ no longer represents a closed contour and it bounds deviations in certain directions only.

Now consider the determinant of $\mathscr{K}'$ which can be written

$$\operatorname{Det}(\mathscr{K}') = 3(\lambda'-\nu')p^2 + \{3(\lambda'-\nu')[4(\mu'-\nu')+(\mu'-\lambda')] - 9(J_1'^2+\tfrac{1}{16}J_3'^2)\}p + \{12(\mu'-\nu')(\mu'-\lambda')(\lambda'-\nu') - 9[J_1'^2(\mu'-\lambda')+\tfrac{1}{4}J_3'^2(\mu'-\nu')]\}$$

Let us define

$$B = 2(\mu'-\nu')+\tfrac{1}{2}(\mu'-\lambda') - [\tfrac{3}{2}(J_1'^2+\tfrac{1}{16}J_3'^2)/(\lambda'-\nu')]$$
$$C = 4(\mu'-\nu')(\mu'-\lambda') - 3[J_1'^2(\mu'-\lambda')+\tfrac{1}{4}J_3'^2(\mu'-\nu')]/(\lambda'-\nu')$$

and let

$$p_1 = -B-(B^2-C)^{1/2}; \quad p_2 = -B+(B^2-C)^{1/2}$$

which represent the roots of $\operatorname{Det}(\mathscr{K}') = 0$. If $\lambda' = \nu'$ the determinant becomes linear in p and has only one root

$$p_3 = (\lambda'-\mu')(J_1'^2+J_3'^2/4)/(J_1'^2+J_3'^2/16)$$

The other root is infinite.

We can now prove the following theorem which identifies the boundaries of regions of qualitatively different stability behavior as p is varied.

Theorem 4: For any given K, ϕ, and ψ the signs of the eigenvalues of $\mathscr{K}'$ are as follows: a) for $\lambda' < \nu'$: for $p < p_1$ three negative eigenvalues, for $p_1 < p < p_2$ two negative and one positive, for $p > p_2$ two positive and one negative; b) for $\lambda' > \nu'$: for $p < p_1$ two negative and one positive, for $p_1 < p < p_2$ two positive and one negative, for $p > p_2$ three positive; c) for $\lambda' = \nu'$: for $p < p_3$ two negative and one positive, for $p > p_3$ two positive and one negative.

The proof of the theorem would follow easily from Lemmas 1 and 2 if we knew that one and only one eigenvalue changes sign each time the determinant goes to zero. Since we don't know this, the arguments become somewhat more complicated.

The determinant of $\mathscr{K}'$ goes to zero at two values of p if the roots p_1 and p_2 are both real and unequal, one value of p if $p_1 = p_2$, one value of p if $\lambda' = \nu'$, and no values of p if p_1 and p_2 are complex. Lemma 2 shows that at least two eigenvalues change sign so there must be at least one real root to $\operatorname{Det}(\mathscr{K}') = 0$. Thus, p_1 and p_2 cannot be complex. Remembering that the determinant is quadratic in p, we note that when $p_1 = p_2$ the sign of the determinant will not change as p passes through p_1. Since the determinant is equal to the product of the eigenvalues, we conclude that either no eigenvalue changes sign or two eigenvalues change sign simultaneously. The former possibility is eliminated by Lemma 2, and now the theorem is easily proved using Lemma 2 for the case $p_1 = p_2$.

Now consider $p_1 \neq p_2$. Then the determinant changes sign as p passes through each value p_1 and p_2. This implies that either one or three eigenvalues change sign at these points. Consider $\lambda' < \nu'$. Lemma 2 shows that for $p < p_1$ we have three negative roots. As p increases into the range $p_1 < p < p_2$ one or three eigenvalues change sign. Use of Lemma 1 and Lemma 2 together form a contradiction in the latter case, so that we get two negative and one positive eigenvalue. Lemma 2 establishes the eigenvalues for $p > p_2$. If $\lambda' > \nu'$ and $p < p_1$ Lemma 2 says there are two negative and one positive eigenvalue. As p passes through p_1 there will be either one or three changes of sign of the eigenvalues. In the former case we could get one negative and two positive, or three positive eigenvalues—a possibility which can be eliminated by Lemma 1. In the latter case we again get one negative and two positive. Lemma 2 gives the result for $p > p_2$.

For $\lambda' = \nu'$ we note that large negative p gives a positive determinant. Thus we have either one positive and two negative, or three positive eigenvalues. Lemma 2 eliminates the latter case by showing that there must be at least two negative eigenvalues. Similar arguments for $p > p_3$ complete the proof of the theorem.

Theorem 4 gives the boundaries of the region of the parameter space which give Liapunov stable equilibria (using the Hamiltonian as a Liapunov function) as $p > p_2$ and $\lambda' > \nu'$. The latter condition needs to be translated into expressions involving ϕ, ψ, and K so that the stable equilibria can be readily visualized

Let us examine the condition $\lambda' = \nu'$ or equivalently $\lambda = \nu$. In terms of the ξ_α matrices it becomes

$$\xi_1{}^T I \xi_1 = \xi_3{}^T I \xi_3$$

which must be satisfied in addition to the equilibrium and orthonormality conditions

$$\xi_1{}^T I \xi_3 = 0, \quad \xi_1{}^T \xi_1 = \xi_3{}^T \xi_3 = 1, \quad \xi_1{}^T \xi_3 = 0$$

The above forms a set of five nonlinear equations in the six unknown direction cosine matrix elements. Combining orthonormality conditions with the first two of the equations gives the altered form

$$(\xi_{11}{}^2 - \xi_{31}{}^2) = -K(\xi_{13}{}^2 - \xi_{33}{}^2); \quad \xi_{11}\xi_{31} = -K\xi_{13}\xi_{33}$$

Here the second subscript refers to the element of the matrix ξ_1 or ξ_3 (i.e., $\xi_{\alpha\beta} = \Theta_{\beta\alpha}$). Squaring the first equation and adding to four times the second and then taking the square root yields

$$(\xi_{11}{}^2 + \xi_{31}{}^2) = K(\xi_{13}{}^2 + \xi_{33}{}^2)$$

where we have used the fact that K is positive for $I_2 > I_3 > I_1$. Combining all three equations we see that

$$\xi_{11} = (\pm)_1 (K)^{1/2} \xi_{33}, \quad \xi_{31} = -(\pm)_1 (K)^{1/2} \xi_{13} \tag{9}$$

with $(\pm)_1$ representing an arbitrary choice of sign.

The sum of the squares of the elements of any row of Θ must equal unity. Multiplying K times this relation for the third row, subtracting the relation for the first row and using Eq. (9) gives

$$\xi_{21}{}^2 - K\xi_{23}{}^2 = (1-K)$$

using Eq. (9) in the cross product relation

$$\xi_{22} = \xi_{11}\xi_{33} - \xi_{31}\xi_{13}$$

gives

$$\xi_{22} = (\pm)_1 (K)^{1/2} (1 - \xi_{23}{}^2)$$

Using these two results in $\xi_2{}^T \xi_2 = 1$ we can solve for ξ_{23} and find that it is identically zero. The elements ξ_{21} and ξ_{22} are now given by the above equations. Then we can set $\xi_{13} = \cos\beta$ and $\xi_{33} = \sin\beta$ where β will turn out to be arbitrary. These can be used in Eq. (9) to get ξ_{11} and ξ_{31}. The remaining elements are obtained from $\xi_1{}^T \xi_1 = \xi_3{}^T \xi_3 = 1$ together with dot products of rows and columns to resolve sign ambiguities. The resulting direction cosine matrix is

$$\Theta = \begin{bmatrix} (\pm)_1 (K)^{1/2} \sin\beta & (\pm)_2 (1-K)^{1/2} & -(\pm)_1 (K)^{1/2} \cos\beta \\ -(\pm)_2 (1-K)^{1/2} \sin\beta & (\pm)_1 (K)^{1/2} & (\pm)_2 (1-K)^{1/2} \cos\beta \\ \cos\beta & 0 & \sin\beta \end{bmatrix}$$

Examination of this matrix shows that $\boldsymbol{\xi}_2$ lies in the $\mathbf{X}_1 - \mathbf{X}_3$ plane. From Fig. 1 it can be seen that this requires $\phi = \theta$ which implies case 4a. Thus case 4a divides those satellites which can be stabilized by picking J_2 large enough from the satellites which are always unstable (in the presence of damping as described above). The 3, 3 component of Θ is $\boldsymbol{\xi}_3 \cdot \mathbf{X}_3$ which shows that the parameter β is actually the angle ψ shown in Fig. 1. Comparison of Θ with ξ_3 from Eq. (3) shows that

$$\cos\phi = \pm(K)^{1/2}$$

which gives four values of ϕ which are symmetric about $\phi = 0$. Now we wish to identify the region for which $\lambda > \nu$. We can specify $\psi = 0$ and η as $\pi/2$ and use Eq. (3) to evaluate λ and ν. We conclude that $\lambda > \nu$ for

$$\begin{gathered} -\cos^{-1}(K)^{1/2} < \phi < \cos^{-1}(K)^{1/2} \\ \pi - \cos^{-1}(K)^{1/2} < \phi < \pi + \cos^{-1}(K)^{1/2} \end{gathered} \tag{10}$$

where principal value is assumed for the inverse function. The satellite orientations which are stable for $p > p_2$ are now easily seen in Fig. 1. Notice that for $K = 0$ the region covers the entire sphere with the boundary line at $\phi = \pm\pi/2$. As K increases, the boundary lines, which can be thought of as longitude lines on the unit sphere, shrink until at $K = 1$ the boundary longitudes coincide at $\phi = 0$ and $\phi = \pi$ and no satellite can be stabilized. Note that $K = 0$ and $K = 1$ imply symmetric satellites. This constitutes a proof of the following stability theorem.

Theorem 5: Given a satellite which is rigid except that it contains one or more constant speed symmetric rotors. Let body axes be chosen so that $I_2 > I_3 > I_1$ (where the inertias include the rotor masses). Let an equilibrium orientation be specified by Eq. (4) and Eq. (5), in terms of the parameters ϕ and ψ and let the total angular momentum vector of the rotor(s) relative to the body of the satellite have components $\omega_0 J_1$ and $\omega_0 J_3$ along ξ_1 and ξ_3, with J_1 and J_3 specified as functions of the equilibrium orientation by Eq. (1). Then any such equilibrium orientation is Liapunov stable if the following conditions are satisfied: a) the longitude ϕ of the geocentric vertical direction ξ_3 is within one of the two regions given by Eq. (10); b) the angular momentum of the rotor relative to the satellite has a component $\omega_0 J_2$ along the perpendicular to the orbital plane which is greater than $\omega_0(I_2 - I_1)p_2$. The parameter ψ is arbitrary. These stability conditions apply to all four cases. Case 4a solutions are never Liapunov stable using the Hamiltonian as a Liapunov function.

The last statement of the theorem follows immediately from Theorem 4. Of course application of Theorem 1 to a satellite containing complete damping as described previously makes the stability conditions of Theorem 5 both necessary and sufficient—in a heuristic sense— for Liapunov stability.

Stability Boundary Plots

The stability region boundaries which depend only on K and ϕ are easily visualized from Fig. 1 as described above. The boundary depending on p is more complicated. We will make a series of contour plots of constant p_α lines as a function of ϕ and ψ for various values of K. This gives a good picture of how the stability boundaries depending on p vary as a function of the other three parameters.

Each choice of ϕ and ψ for a given K corresponds to two equilibrium orientations resulting from the two solutions for η from Eq. (5). In Fig. 1 we see that these two solutions result from the fact that the positive sense of the $\boldsymbol{\xi}_1$ axis is arbitrary. If we change the sign of ξ_1 then we must change the sign of ξ_2 to preserve a right-handed system. By Eq. (1), λ, μ, ν, and J_1 are unaffected in this process while J_3 changes sign. Since the characteristic equation for $\mathscr{H}$ remains unaltered we conclude that the stability properties of the two solutions are identical.

The range of the values for ϕ and ψ can be taken as $-\pi < \phi < \pi$, and $-\pi/2 < \psi < \pi/2$. However, because of certain symmetries we need only consider ϕ and ψ between 0 and $\pi/2$. To show this, consider changing ϕ to $-\phi$ which requires

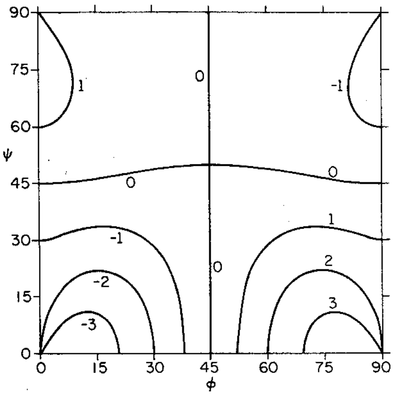

Fig. 2 Constant p_- contours for $K = \frac{1}{2}$.

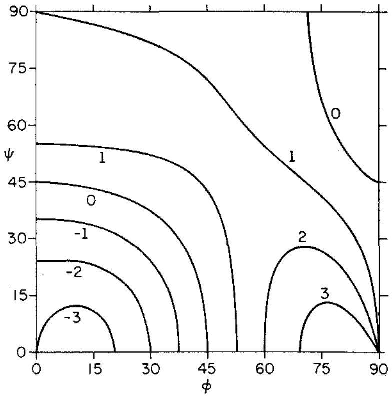

Fig. 4 Constant p_- contours for $K = \frac{1}{4}$.

changing θ to $-\theta$ [Eq. (4)] while η and ψ remain unchanged. Using Eq. (3) in Eq. (1) shows that λ, μ, and ν do not change sign and J_1 and J_3 do. Then the determinant of $\mathscr{K}'$ remains unaltered so that the stability boundaries p_1 and p_2 are symmetric about $\phi = 0$. Similar arguments show that the stability boundaries are also symmetric about $\psi = 0$.

The contour plots are slightly easier to interpret if we plot roots p_+ and p_- as defined here, rather than p_1 and p_2

$$p_+ = [-b+(b^2-4ac)^{1/2}]/2a; \quad p_- = [-b-(b^2-4ac)^{1/2}]/2a$$

Here a and b are coefficients of p^2 and p in the determinant of $\mathscr{K}'$, and c is the term independent of p. Since $a = 3(\lambda'-\nu')$, we see that as ϕ passes through $\cos^{-1}(K)^{1/2}$ this coefficient goes through zero, and p_+ passes through the point at infinity. Thus as we approach the boundary $\phi = \theta$, the p required to make the satellite Liapunov stable (using the Hamiltonian as a Liapunov function) tends toward infinity.

Figure 2 shows lines of constant p_- in the ϕ, ψ plane for $K = \frac{1}{2}$. Figure 3 gives corresponding contours for p_+. Note that all the contours seem to cross the ϕ and ψ axes in a direction perpendicular to the axes, except at the corners of the plots. This might have been anticipated because of the symmetries in ϕ and ψ noted above. One region of Liapunov stable equilibrium orientations for $K = \frac{1}{2}$ has $-45° < \phi < 45°$ and $p > p_+$. Figure 3 shows this region for ϕ and ψ positive, and reflecting about each axis gives the p_+ contours for the remainder of the region. The other region of stable equilibria is identical except that ϕ is shifted by 180°. Note that there is an area between $\phi = 0$ and $\phi = 15°$ for which p_+ is negative so that these satellites are Liapunov stable even if the component of rotor angular momentum along ξ_2 is zero or slightly negative. It should be noted that if you approach the origin $\phi = \psi = 0$ along $\psi = 0$ the orientation you obtain corresponds to the well known stable equilibrium for a rigid satellite, which has the axis of least moment of inertia pointing toward the earth and the axis of greatest moment of inertia along the perpendicular to the orbital plane. Since this orientation is stable without a rotor we know it must be contained within the $p_+ = 0$ contour.

From Theorem 4, values of p between p_+ and p_- with $\phi < 45°$ gives two positive and one negative eigenvalue to the $\mathscr{K}'$ matrix. This corresponds to three dimensional "saddle points" in the dynamic potential for these orientations. With p less than p_- we get two negative and one positive eigenvalue. When $\phi > 45°$, which means that $\lambda' < \nu'$, the regions with various numbers of negative eigenvalues can easily be identified. The equilibria occur at local maxima of the dynamic potential only if $\phi > 45°$ and $p < p_+$.

Figures 4 and 5 show lines of constant p_- and p_+ for $K = \frac{1}{4}$. Comparison with Figs. 2 and 3 gives some idea of the behavior of the surfaces as K is changed. The boundary $\theta = \psi$ occurs at 60° so that the stability region is enlarged. The $p_+ = 0$ contour has become significantly smaller, and generally it takes a larger value of p to be in the region of stable equilibria. Figure 6 gives the constant p_- and p_+ lines when $K = 0$ which indicates that the satellite is inertially symmetric. This implies that all of the orientations belong to cases 1, 2, or 3. Note that the $p_+ = 0$ contour has collapsed to the origin.

All possible satellite inertias are obtained when we vary K from zero to unity. Thus it would be desirable to have plots for $K = \frac{3}{4}$ and $K = 1$ to give a complete picture of the behavior of the stability boundaries. We will show that Figs. 4, 5, and 6, when properly interpreted, give the desired plots. Consider the transformation where K is changed to $1-K$, ϕ to $\pi/2-\phi$, and θ to $\pi/2-\theta$. Equations (4) and (5) are invariant under this transformation. Writing J_1', J_3', λ', μ', and ν' in terms of the angles ϕ, ψ, θ, and η shows that J_1' and J_3' are also invariant while the differences $(\mu'-\nu')$, $(\lambda'-\nu')$, and $(\mu'-\lambda')$ all change sign under the transformation. If we include one further transformation, changing p to $-p$, then it is easily seen that the equation obtained by setting the determinant of $\mathscr{K}'$ equal to zero is also invariant. Thus the boundaries p_- and p_+ change sign. We conclude that Figs. 4 and 5 can be read for $K = \frac{3}{4}$, and Fig. 6 for $K = 1$, if ϕ is changed to $(90°-\phi)$ and the signs of the constants of the contour lines are changed. Note that the area within the $p_+ = 0$ contour near the origin decreases slightly and in the ϕ direction and increases significantly in the ψ direction as K is increased from $\frac{1}{2}$ to $\frac{3}{4}$.

Although the case 4 solutions are of primary interest here,

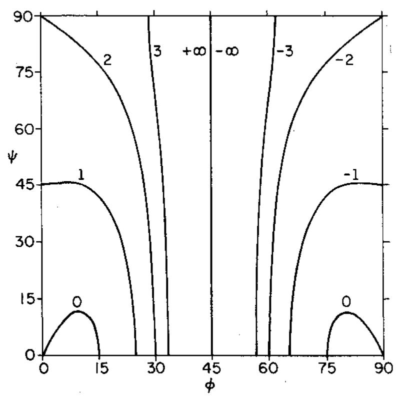

Fig. 3 Constant p_+ contours for $K = \frac{1}{2}$.

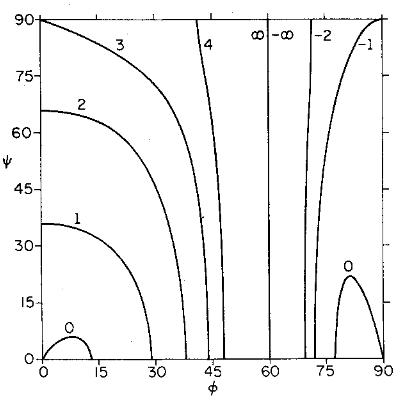

Fig. 5 Constant p_+ contours for $K = \frac{1}{4}$.

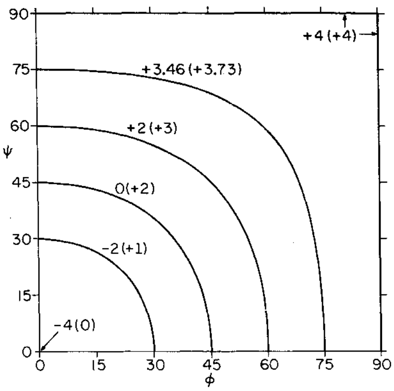

Fig. 6 Constant p_- and p_+ contours for symmetric satellites with $K = 0$. Values of p_+ occur in parentheses.

it is interesting to see where the elementary cases 1, 2, and 3 appear on the ϕ, ψ plots. Case 2 orientations require a body principal axis to be aligned with $\pm\xi_1$. From Eqs. (4) and (5) and Fig. 1 we see that this happens along the vertical axes $\phi = 0$ and $\phi = 90°$, and along the horizontal axis $\psi = 0$. Case 3 orientations occur when a principal axis is aligned with $\pm\xi_3$ and the other two principal axes can lie anywhere in the $\xi_1 - \xi_2$ plane, depending on the rotor speed and orientation within the satellite. Examination of Fig. 1 shows that all case 3 orientations occur when $\phi = \psi = 0$, or $\psi = 0$ and $\phi = 90°$, or $\psi = 90°$ with ϕ arbitrary. The last point mentioned has been expanded in the contour plots because of the projection from the unit sphere onto rectangular axes. However, the other two points have not been expanded and thus the origin and $\psi = 0$, and $\phi = 90°$ on the contour plots are singular points in the sense that the values of p_+ and p_- obtained in the limit as you approach these points depends on the direction of approach. Each direction ends on a different case 3 (or case 1) orientation for the satellite. Case 1 orientations are obtained by approaching any of the corners of the contour plots along one of the axes. Thus, the origin represents two case 1 solutions and a set of case 3 solutions.

In the special cases the values of p_+ and p_- can be obtained directly from the $\mathcal{K}'$ matrix without use of the quadratic formula, since J_1 and/or J_3 are zero (Ref. 4 gives the results). Substituting case 3 orientation for $\phi = \psi = 0$ we find that the two roots p_+ and p_- are $K \cos(2\eta)$ and

$$-\{4(1-K) + K^2 \sin^2\eta \cos^2\eta\}/(1 - K \sin^2\eta)$$

(which order depends on the size of K). The angle η can be visualized on the plot as the angle of approach to the origin measured from the vertical. Making use of the invariants noted above we can obtain the corresponding values for $\phi = 90°$, $\psi = 0$ by changing the signs and transforming K to $1-K$. We see that these two singular points take on only finite values, and when visualized in three dimensions they resemble a portion of the center of a spiral. Because of the small number of contours taken in the contour plots the above formulas are necessary to get a good picture of the behavior near the singular points. However, the plots do show that the behavior near the singular point $\psi = 90°$ is somewhat different in that infinite values are assumed for p_+ when $\phi = \theta$.

Simple expressions for p_+ and p_- can be obtained as above for all the edges of the contour plots. This fact was used to obtain Fig. 6. For $K = 0$ the $\mathbf{X}_1$ body axis is an axis of symmetry. In the approach taken in this paper the rotor has not been fixed within the body so that the contours in Fig. 6 must represent a pure rotation of the body about $\mathbf{X}_1$. Starting directly from Eq. (8) the case 2 orientations along the axis $\phi = 0$ give roots $p_+ = 4\sin^2\psi$ and $p_- = -4\cos 2\psi$. If ψ^* represents any value of ψ along this axis, the values of ψ and ϕ after rotation are related by $\cos\psi^* = \cos\phi\cos\psi$. Combining these results gives Fig. 6. One further point should be mentioned about this figure. As K approaches zero the $p_+ = \infty$ contour approaches $\phi = 90°$. However, when K reaches zero we get limiting values of $p_+ = 4$ and the infinite values have disappeared. Because of the symmetry about $\mathbf{X}_1$ the singular point at $\phi = 90°$, $\psi = 0$ spreads into a singular line along $\phi = 90°$ with case 3 solutions for which $p_+ = \sin^2\eta$ and $p_- = -\cos 2\eta$. This band of values also continues along $\psi = 90°$.

Conclusions

The two parameter family of equilibrium orientations discussed here, gives a wide choice in the design of gravitationally stabilized satellites. Any configuration in the Liapunov stable region described previously might be considered in preliminary satellite design. Because of the large choice among the design parameters one could find an orientation which had optimal stability properties for any particular application. No general optimum could be found since the optimum depends on the damping mechanism employed, but for each application one could vary the parameters to obtain the shortest damping time for the expected perturbations. The parameters would include any constants associated with the damping mechanisms plus $I_2 - I_1$, K, ϕ, ψ, and J_2.

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